Computable Fourier Conditions for Alias-Free Sampling and Critical Sampling

Yue M. Lu [(1)(2)], Minh N. Do [(2)] and Richard S. Laugesen [(2)]

(1) Ecole Polytechnique Fédérale de Lausanne (EPFL), CH-1015, Lausanne, Switzerland
(2) University of Illinois at Urbana-Champaign, Urbana IL 61801, USA
yue.lu@epfl.ch, minhdo@illinois.edu, Laugesen@illinois.edu

Abstract:

We propose a Fourier analytical approach to the problems of alias-free sampling and critical sampling. Central to this approach are two Fourier conditions linking the above sampling criteria with the Fourier transform of the indicator function defined on the underlying frequency support. We present several examples to demonstrate the usefulness of the proposed Fourier conditions in the design of critically sampled multidimensional filter banks. In particular, we show that it is impossible to implement any cone-shaped frequency partitioning by a nonredundant filter bank, except for the 2-D case.

1 Introduction

The search for *alias-free sampling* lattices for a given frequency support, and in particular for those lattices achieving minimum sampling densities, is a fundamental issue in various signal processing applications that involve the design of efficient acquisition schemes for bandlimited signals. As a special case of alias-free sampling, the concept of *critical sampling* also plays an important role in the theory and design of critically sampled (a.k.a. maximally decimated) multidimensional filter banks [9].

The study of alias-free (and critical) sampling lattices is a classical problem [8, 4]. So far, most existing work in the literature approaches the problem from a geometrical perspective: The primary tools employed include the theories from Minkowski's work [2], as well as various geometrical intuitions and heuristics.

In this paper, we propose a Fourier analytical approach to the problems of alias-free sampling and critical sampling. Central to this approach are two Fourier conditions linking the above sampling criteria with the Fourier transform of the indicator function defined on the underlying frequency support (see Theorem 1 and Proposition 2). An important feature of the proposed conditions is that they open the door to purely analytical and computational solutions to the sampling lattice selection problem.

The rest of the paper is organized as follows. In Section 2, we briefly review some relevant concepts on sampling bandlimited signals. We present in Section 3 a novel condition linking the alias-free sampling (as well as critical sampling) with the Fourier transform of the indicator function defined on the given frequency support. In Section 4, we present an application of the proposed Fourier conditions in the design of multidimensional nonredundant filter banks. We conclude the paper in Section 5. The material in this paper was presented in part in [5] and [7]. As a novel aspect, we present in this paper a different proof for Theorem 1, which provides important new insights into this key result.

Notation: The Fourier transform of a function $f(\boldsymbol{\omega})$ defined on $\mathbb{R}^N$ is defined by

$$\widehat{f}(\boldsymbol{x}) = \int_{\mathbb{R}^N} f(\boldsymbol{\omega})\, e^{-2\pi j\, \boldsymbol{x}\cdot\boldsymbol{\omega}}\, d\boldsymbol{\omega}. \tag{1}$$

Calligraphic letters, such as $\mathcal{D}$, represent bounded and open frequency domains in $\mathbb{R}^N$, with $m(\mathcal{D})$ denoting the Lebesgue measure (*i.e.* volume) of $\mathcal{D}$. Given a nonsingular matrix $\boldsymbol{M}$ and a vector $\boldsymbol{\tau}$, we use $\boldsymbol{M}(\mathcal{D}+\boldsymbol{\tau})$ to represent the set of points of the form $\boldsymbol{M}(\boldsymbol{\omega}+\boldsymbol{\tau})$ for $\boldsymbol{\omega}\in\mathcal{D}$. Finally, we denote by $\mathbb{1}_{\mathcal{D}}(\boldsymbol{\omega})$ the indicator function of the domain $\mathcal{D}$, *i.e.*, $\mathbb{1}_{\mathcal{D}}(\boldsymbol{\omega}) = 1$ if $\boldsymbol{\omega}\in\mathcal{D}$ and $\mathbb{1}_{\mathcal{D}}(\boldsymbol{\omega}) = 0$ otherwise.

2 Background

In multidimensional multirate signal processing, the sampling operations are usually defined on lattices, each of which can be generated by an $N\times N$ nonsingular matrix $\boldsymbol{M}$ as

$$\Lambda_{\boldsymbol{M}} \stackrel{\text{def}}{=} \{\boldsymbol{M}\boldsymbol{n} : \boldsymbol{n}\in\mathbb{Z}^N\}. \tag{2}$$

We denote by $\Lambda^*_{\boldsymbol{M}}$ the corresponding reciprocal lattice (a.k.a. polar lattice), defined as

$$\Lambda^*_{\boldsymbol{M}} \stackrel{\text{def}}{=} \{\boldsymbol{M}^{-T}\boldsymbol{\ell} : \boldsymbol{\ell}\in\mathbb{Z}^N\} \tag{3}$$

In the rest of the paper, when it is clear from the context what the generating matrix is, we will drop the subscripts in $\Lambda_{\boldsymbol{M}}$ and $\Lambda^*_{\boldsymbol{M}}$, and use Λ and Λ^* for simplicity.

Let $f(\boldsymbol{x})$ be a continuous-domain signal, whose Fourier transform is bandlimited to a bounded open set $\mathcal{D}\subset\mathbb{R}^N$. The discrete-time Fourier transform of the samples $s[\boldsymbol{n}] \stackrel{\text{def}}{=} f(\boldsymbol{M}\boldsymbol{n})$ is supported in [9]

$$\mathcal{S} = \boldsymbol{M}^T \left(\bigcup_{\boldsymbol{k}\in\Lambda^*} (\mathcal{D}+\boldsymbol{k}) \right). \tag{4}$$

For appropriately chosen sampling lattices, the aliasing components in (4) do not overlap with the baseband frequency support $\mathcal{D}$. In this important case, we can fully recover the original continuous-domain signal $f(\boldsymbol{x})$ by applying an ideal interpolation filter spectrally supported on $\mathcal{D}$ to the discrete samples $s[\boldsymbol{n}]$.

Definition 1 *We say a frequency support $\mathcal{D}$ allows an* alias-free *$\boldsymbol{M}$-fold sampling, if different shifted copies of $\mathcal{D}$ in* (4) *are disjoint, i.e.,*

$$\mathcal{D} \cap (\mathcal{D} + \boldsymbol{k}) = \emptyset \ \text{ for all } \ \boldsymbol{k} \in \Lambda^* \setminus \{\boldsymbol{0}\}\,. \tag{5}$$

Furthermore, we say $\mathcal{D}$ can be critically sampled *by $\boldsymbol{M}$, if in addition to the alias-free condition in* (5)*, the union of the shifted copies also covers the entire spectrum, i.e.,*

$$\bigcup_{\boldsymbol{k}\in\Lambda^*} (\mathcal{D} + \boldsymbol{k}) = \mathbb{R}^N, \quad \text{up to a set of measure zero}. \tag{6}$$

The focus of this work is to present two Fourier analytical conditions for alias-free sampling and critical sampling. Our discussions will be based on the following geometrical argument [2], which can be easily verified from (5).

Proposition 1 *The alias-free sampling condition in* (5) *is equivalent to requiring*

$$\Lambda^* \cap (\mathcal{D} - \mathcal{D}) = \{\boldsymbol{0}\}\,, \tag{7}$$

where $\mathcal{D} - \mathcal{D} \overset{def}{=} \{\boldsymbol{\omega} - \boldsymbol{\tau} : \boldsymbol{\omega}, \boldsymbol{\tau} \in \mathcal{D}\}$ is the Minkowski sum of the open set $\mathcal{D}$ and its negative $-\mathcal{D}$.

3 Fourier Analytical Conditions

In this section, we study the problems of alias-free sampling and critical sampling with Fourier techniques. The key observation is a link between the alias-free sampling condition and the Fourier transform of the indicator function $\mathbb{1}_{\mathcal{D}}(\boldsymbol{\omega})$ defined on the frequency support $\mathcal{D}$.

31 Alias-Free Sampling

Lemma 1 *Let $\mathcal{D}$ be a frequency region, and $f(\boldsymbol{\omega})$ a positive function supported on $(\mathcal{D} - \mathcal{D})$, i.e., $f(\boldsymbol{\omega}) > 0$ for $\boldsymbol{\omega} \in (\mathcal{D} - \mathcal{D})$ and $f(\boldsymbol{\omega}) = 0$ otherwise. Then $\mathcal{D}$ allows an $\boldsymbol{M}$-fold alias-free sampling* if and only if

$$\sum_{\boldsymbol{k}\in\Lambda^*} f(\boldsymbol{k}) = f(\boldsymbol{0}). \tag{8}$$

Proof By construction, (8) holds if and only if $\Lambda^* \cap (\mathcal{D} - \mathcal{D}) = \{\boldsymbol{0}\}$. Applying Proposition 1, we are done. ∎

Theorem 1 *A frequency region $\mathcal{D}$ allows an $\boldsymbol{M}$-fold alias-free sampling* if and only if

$$|\boldsymbol{M}| \sum_{\boldsymbol{n}\in\Lambda} |\widehat{\mathbb{1}}_{\mathcal{D}}(\boldsymbol{n})|^2 = m(\mathcal{D}), \tag{9}$$

where $\widehat{\mathbb{1}}_{\mathcal{D}}(\boldsymbol{x})$ is the Fourier transform of $\mathbb{1}_{\mathcal{D}}(\boldsymbol{\omega})$, and $|\boldsymbol{M}|$ is the absolute value of the determinant of $\boldsymbol{M}$.

Proof Consider the autocorrelation function

$$R_{\mathcal{D}}(\boldsymbol{\omega}) = \int \mathbb{1}_{\mathcal{D}}(\boldsymbol{\tau})\ \mathbb{1}_{\mathcal{D}}(\boldsymbol{\tau} - \boldsymbol{\omega})\, d\boldsymbol{\tau}.$$

Clearly, $R_{\mathcal{D}}(\boldsymbol{\omega}) \geq 0$ for all $\boldsymbol{\omega}$. Meanwhile, we can verify that supp $R_{\mathcal{D}}(\boldsymbol{\omega}) = (\mathcal{D} - \mathcal{D})$. Thus, we can apply Lemma 1 and obtain that, $\mathcal{D}$ allows an $\boldsymbol{M}$-fold alias-free sampling if and only if

$$\sum_{\boldsymbol{k}\in\Lambda^*} R_{\mathcal{D}}(\boldsymbol{k}) = R_{\mathcal{D}}(\boldsymbol{0}) = \int \mathbb{1}_{\mathcal{D}}(\boldsymbol{\tau})\, d\boldsymbol{\tau} = m(\mathcal{D}).$$

Applying the Poisson summation formula to the above equality (see Appendix A of [7] for a justification of the pointwise equality), we have

$$m(\mathcal{D}) = \sum_{\boldsymbol{k}\in\Lambda^*} R_{\mathcal{D}}(\boldsymbol{k}) = |\boldsymbol{M}| \sum_{\boldsymbol{n}\in\Lambda} \widehat{R}_{\mathcal{D}}(\boldsymbol{n}). \tag{10}$$

From the definition of $R_{\mathcal{D}}(\boldsymbol{\omega})$, its Fourier transform is $\widehat{R}_{\mathcal{D}}(\boldsymbol{x}) = |\widehat{\mathbb{1}}_{\mathcal{D}}(\boldsymbol{x})|^2$. Substituting this formula into (10), we are done. ∎

32 Critical Sampling

Here we focus on the special case of critical sampling, and begin by mentioning, without proof, a standard result:

Lemma 2 *A frequency support $\mathcal{D}$ can be critically sampled by a sampling matrix $\boldsymbol{M}$ if and only if $\boldsymbol{M}$ is an alias-free sampling matrix for $\mathcal{D}$ with sampling density $1/|\boldsymbol{M}| = m(\mathcal{D})$.*

Proposition 2 *A frequency support $\mathcal{D}$ can be critically sampled by a matrix $\boldsymbol{M}$ if and only if*

$$\widehat{\mathbb{1}}_{\mathcal{D}}(\boldsymbol{0}) = m(\mathcal{D}) = \frac{1}{|\boldsymbol{M}|} \quad \text{and} \quad \widehat{\mathbb{1}}_{\mathcal{D}}(\boldsymbol{n}) = 0 \tag{11}$$

for all $\boldsymbol{n} \in \Lambda \setminus \{\boldsymbol{0}\}$.

Proof Suppose (11) holds. Then it follows that

$$\sum_{\boldsymbol{n}\in\Lambda} |\widehat{\mathbb{1}}_{\mathcal{D}}(\boldsymbol{n})|^2 = |\widehat{\mathbb{1}}_{\mathcal{D}}(\boldsymbol{0})|^2 = \frac{m(\mathcal{D})}{|\boldsymbol{M}|},$$

and hence from Theorem 1, $\boldsymbol{M}$ is an alias-free sampling matrix for $\mathcal{D}$. Meanwhile, since $m(\mathcal{D}) = \frac{1}{|\boldsymbol{M}|}$, we can apply Lemma 2 to conclude that $\mathcal{D}$ is critically sampled by $\boldsymbol{M}$. By reversing the above line of reasoning, we can also show the necessity of (11). ∎

Remark: The result of Proposition 2 is previously known in various disciplines. In approximation theory, the condition (11) is often called the interpolation property (see, for example, [4]). The usefulness of this condition in the context of lattice tiling was first pointed out by Kolountzakis and Lagarias [3] and applied to investigate the tiling of various high dimensional shapes.

33 Computational Aspects

The Fourier conditions proposed in Theorem 1 and Proposition 2 can lead to practical computational algorithms for testing alias-free and critical sampling. Here, we briefly comment on two important computational aspects in applying the proposed conditions.

First, as a prerequisite to using the proposed Fourier conditions, we must know the expression for $\widehat{\mathbb{1}}_{\mathcal{D}}(\boldsymbol{x})$. This evaluation can be a cumbersome task if we need to do the derivation by hand for each given $\mathcal{D}$. However, when the frequency regions $\mathcal{D}$ are arbitrary polygonal and polyhedral domains, we can obtain the closed-form expressions for $\widehat{\mathbb{1}}_{\mathcal{D}}(\boldsymbol{x})$ via the divergence theorem [1, 7].

Another potential issue in practical implementations is that the Fourier conditions in (9) and (11) both involve an infinite number of lattice points. We show in [7] that the infinite sum in (9) can be well-approximated by a truncated finite sum. Moreover, with high probability, we actually only need to evaluate the Fourier transform on a very small number of points in a lattice (e.g. 4 points in 2-D) in order to show aliasing occurs, thus ruling out the lattice.

4 Application: Filter Bank Design

In this section we present an application of Proposition 2 in the design of multidimensional critically sampled filter banks.

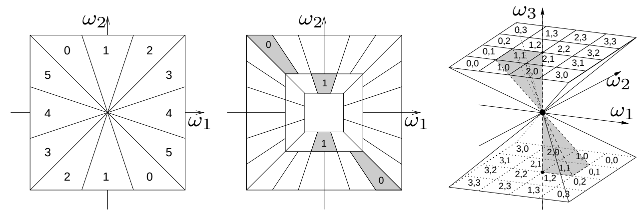
41 Frequency Partitioning of Critically Sampled Filter Banks

Consider a general multidimensional filter bank, where each channel contains a subband filter and a sampling operator. As an important step in filter bank design, we need to specify the ideal passband support of each subband filter, all of which form a partitioning of the frequency spectrum.

Not every possible frequency partitioning can be used for filter bank implementation though. In particular, if we want to have a nonredundant filter bank, then the ideal passband support of each subband filter must be critically sampled by the sampling matrix in that channel. Consequently, whenever given a possible frequency partitioning, we must first perform a "reality check" of seeing whether the above condition is met, before proceeding to actual filter design.

The critical sampling condition is commonly verified geometrically (*i.e.* by drawing figures). Although intuitive and straightforward, this geometrical approach becomes cumbersome when the shape of the passband support is complicated, or when we work in 3-D and higher dimensional cases. Applying the result of Proposition 2, we propose in the following a computational procedure, which can systematically check and determine the critical sampling matrices of a given polytope region. Notice that the algorithm only searches among integer matrices, since the filter banks considered here operate on discrete-time signals.

Procedure 1 *Let $\mathcal{D}$ be a given polytope-shaped frequency support region.*



(a) (b) (c)

Figure 1: The ideal frequency partitioning of several filter banks. (a) A directional filter bank which decomposes the frequency cell $(-\frac{1}{2}, \frac{1}{2}]^2$ into 6 subbands. (b) A directional multiresolution frequency partitioning. (c) A 3-D directional frequency decomposition with pyramid-shaped passband supports.

1. *Calculate $\delta = 1/m(\mathcal{D})$. From* (11)*, any matrix $\boldsymbol{M}$ that can critically-sample $\mathcal{D}$ must satisfy $|\boldsymbol{M}| = \delta$. If δ is not an integer, then stop the procedure, since in this case it is impossible for $\mathcal{D}$ to be critically sampled by any integer matrix.*
2. *Construct a closed-form formula [7] for $\widehat{\mathbb{1}}_{\mathcal{D}}(\boldsymbol{x})$.*
3. *Based on the Hermite normal form, construct an exhaustive list of matrices of determinant δ, each corresponding to a distinct sampling lattice [7].*
4. *For every matrix $\boldsymbol{M}$ in the above list, test the following condition*

$$\widehat{\mathbb{1}}_{\mathcal{D}}(\boldsymbol{M}\boldsymbol{n}) = 0 \quad \textit{for all } \boldsymbol{n} \in \mathbb{Z}^N \setminus \{\boldsymbol{0}\} \textit{ with } \|\boldsymbol{n}\|_\infty \le r, \tag{12}$$

where r is a large positive integer.

5. *Present all the matrices in the list that satisfy* (12)*. If there is no such matrix, then $\mathcal{D}$ cannot be critically sampled by any integer matrix.*

To be clear, the expression (12) is a necessary condition for $\mathcal{D}$ to be critically sampled by $\boldsymbol{M}$. It is not sufficient since we only check for integer points within a finite radius r, and so in principle, even if $\boldsymbol{M}$ satisfies (12) for all $\|\boldsymbol{n}\|_\infty \le r$, it might happen that $\widehat{\mathbb{1}}_{\mathcal{D}}(\boldsymbol{M}\boldsymbol{n}) \neq 0$ for some $\boldsymbol{n}$ with $\|\boldsymbol{n}\|_\infty > r$. However, by choosing r sufficiently large, we can gain confidence in the validity of the original infinite condition (11) as required in Proposition 2. We leave the quantitative analysis of this approximation to [7]. In the following examples, we choose $r = 10000$.

Example 1 *Figure 1(a) presents the frequency decomposition of a directional filter bank (DFB). Applying the algorithm in Procedure 1, we can easily verify that this frequency decomposition can be critically sampled. The corresponding sampling matrices, denoted by $\boldsymbol{M}_k$ for the kth subband, are*

$$\boldsymbol{M}_0 = \boldsymbol{M}_1 = \boldsymbol{M}_2 = \begin{pmatrix} 6 & 3 \\ 0 & 1 \end{pmatrix}.$$

$\boldsymbol{M}_3, \boldsymbol{M}_4$ and $\boldsymbol{M}_5$ can be inferred by symmetry.

Example 2 *We show in Figure 1(b) a directional and multiresolution decomposition of the 2-D frequency spectrum. Applying Procedure 1 confirms that such a frequency partitioning can be critically sampled as well. The sampling* *matrices for two representative subbands (marked as dark regions in the figure) are*

$$\boldsymbol{M}_0 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \text{ and } \boldsymbol{M}_1 = \begin{pmatrix} 8 & 4 \\ 0 & 4 \end{pmatrix}.$$

Example 3 *Figure 1(c) shows an extension of the original 2-D DFB to the 3-D case [6]. Applying Procedure 1, we find that the 3-D frequency partitioning shown in Figure 1(c) cannot be critically sampled; in other words, redundancy is unavoidable for a 3-D DFB.*

42 Critical Sampling of General Cone-Shaped Frequency Regions in Higher Dimensions

The result in Example 3 can be generalized to higher dimensions, and to cases where the subbands take different directional shapes. As an application of the Fourier condition in Proposition 2, we show here a much more general statement: it is impossible to implement any cone-shaped frequency partitioning by a nonredundant filter bank, except for the 2-D case.

We consider the following ideal subband supports in N-D:

$$\mathcal{D} = \{\boldsymbol{\omega} : a \leq |\omega_N| \leq b,\ (\omega_1, \ldots, \omega_{N-1}) \in \omega_N\, \mathcal{B}\}, \quad (13)$$

where $\mathcal{B}$ is some bounded set in $\mathbb{R}^{N-1}$. Geometrically, $\mathcal{D}$ takes the form of a two-sided cone in $\mathbb{R}^N$, truncated by hyperplanes $|\omega_N| = a$ and $|\omega_N| = b$, where $0 \leq a < b$. The "base" region $\mathcal{B}$ in (13) is the intersection between the cone and the hyperplane $\omega_N = 1$.

The formulation in (13) is flexible enough to characterize, up to a rotation, any directional subband shown in Figure 1. For example, the 3-D pyramid-shaped subband $(1, 1)$ in Figure 1(c) can be presented by $a = 0, b = \frac{1}{2}$, and $\mathcal{B} = [-\frac{1}{2}, 0]^2$. However, the class of frequency shapes that can be described by (13) is far beyond those shown in Figure 1, since the formulation (13) allows for arbitrary configuration of the cross section heights a and b (not necessarily the dyadic decomposition as in Figure 1(b)) and arbitrary shape for the base $\mathcal{B}$ (not necessarily lines or squares).

Lemma 3 *If a frequency support $\mathcal{D}$ can be critically sampled by an* integer *matrix $\boldsymbol{M}$, then*

$$\widehat{\mathbb{1}}_{\mathcal{D}}(|\boldsymbol{M}|\, \boldsymbol{n}) = 0, \quad \textit{for all } \boldsymbol{n} \in \mathbb{Z}^N \setminus \{0\}. \quad (14)$$

Proof It is easy to verify that, for any integer matrix $\boldsymbol{M}$, the vector $|\boldsymbol{M}|\, \boldsymbol{n}$ belongs to the lattice Λ generated by $\boldsymbol{M}$. The condition (14) then follows from (11) in Proposition 2. ∎

Theorem 2 *For arbitrary choice of $0 \leq a < b$ and the base shape $\mathcal{B}$, the frequency domain support $\mathcal{D}$ given in* (13) *cannot be critically sampled by any integer matrix in N-dimensions, $N \geq 3$.*

Remark: For 2-D, we established the positive result in Examples 1 and 2.

Proof We argue by contradiction. Suppose for $N \geq 3$, and for some particular choices of $0 \leq a < b$ and $\mathcal{B}$, the corresponding frequency region $\mathcal{D}$ in (13) can be critically sampled by an integer matrix $\boldsymbol{M}$. It then follows from (14) in Lemma 3 that

$$\widehat{\mathbb{1}}_{\mathcal{D}}(0, \ldots, 0, |\boldsymbol{M}|\, n) = 0, \quad \text{for all } n \in \mathbb{Z} \setminus \{0\}. \quad (15)$$

From the definition of $\mathcal{D}$, we have

$$\begin{aligned}
&\widehat{\mathbb{1}}_{\mathcal{D}}(0, \ldots, 0, x) \\
&= \int_{a \leq |\omega_N| \leq b} d\omega_N \left(e^{-2\pi j\, x\, \omega_N} \int_{\omega_N \mathcal{B}} 1\, d\omega_1 \ldots d\omega_{N-1} \right) \\
&= \int_{a \leq |\omega| \leq b} e^{-2\pi j\, x\, \omega}\, m(\omega\, \mathcal{B})\, d\omega \\
&= \int_{a \leq |\omega| \leq b} e^{-2\pi j\, x\, \omega} |\omega|^{N-1} m(\mathcal{B})\, d\omega \\
&= 2\, m(\mathcal{B}) \int_a^b \omega^{N-1} \cos(2\pi x\, \omega)\, d\omega.
\end{aligned}$$

After a change of variable, we can now rewrite (15) as $\int_{2\pi|\boldsymbol{M}|a}^{2\pi|\boldsymbol{M}|b} \omega^{N-1} \cos(n\, \omega)\, d\omega = 0$, for all $n \in \mathbb{Z} \setminus \{0\}$, which is impossible when $N \geq 3$ by Appendix C of [7]. ∎

5 Conclusions

By linking the alias-free (and critical) sampling of a given frequency support region with the Fourier transform of the indicator function, we presented two simple yet powerful conditions for checking alias-free sampling and critical sampling. We demonstrated the usefulness of the proposed conditions in the design of multidimensional critically sampled filter banks. As an interesting result, we show that it is impossible to construct a *nonredundant* directional filter bank with a general cone-shaped frequency decomposition, except for the 2-D case.

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