Sparsity according to Prony, average performance analysis

Jon Oñativia, Pier Luigi Dragotti
Commmunications and Signal Processing Group (CSP),
Imperial College London, London SW7 2AZ, UK
Email: {jon.onativia,p.dragotti}@imperial.ac.uk

Yue M. Lu
Signals, Information, and Networks Group (SING),
Harvard University, Cambridge, MA 02138, USA
Email: yuelu@seas.harvard.edu

***Abstract*—Finding the sparse representation of a signal in an overcomplete dictionary has attracted a lot of attention over the past years. Traditional approaches such as Basis Pursuit are based on relaxing a nonconvex ℓ_0-minimization problem [1]–[3]. In [4], a new polynomial complexity algorithm, ProSparse, is presented. ProSparse solves the sparse representation problem when the dictionary is the union of Fourier and canonical bases and can be extended to other relevant pairs of bases or frames. Here, we present a probabilistic average-case analysis that characterizes a sharp phase transition behaviour of the algorithm. We also present an extension of the algorithm for the noisy scenario. This proposed extension outperforms the Basis Pursuit Denoise algorithm in support retrieval in a number of scenarios.**

I. Introduction

Let $\boldsymbol{y} \in \mathbb{C}^N$ be a complex-valued finite dimensional signal that has a K-sparse representation in an overcomplete dictionary, that is, $\boldsymbol{y} = \boldsymbol{D}\,\boldsymbol{x}$, where $\boldsymbol{D} \in \mathbb{C}^{N\times L}$ is the overcomplete dictionary with $L > N$ atoms and $\boldsymbol{x} = (x_\ell)_{\ell=0}^{L-1} \in \mathbb{C}^L$ satisfies $\|\boldsymbol{x}\|_0 \stackrel{\text{def}}{=} \#\{\ell : |x_\ell| \neq 0\} = K$. When the dictionary consists of the union of Fourier and identity matrices, that is, $\boldsymbol{D} = [\boldsymbol{F}, \boldsymbol{I}]$, the signal $\boldsymbol{y}$ can be written as the sum of K_f Fourier atoms and K_s spikes, where $K_f + K_s = K$.

ProSparse [4] is based on finding *clean* windows of consecutive samples of the signal $\boldsymbol{y}$ where there is no contribution due to the spikes, and therefore the samples are only due to the Fourier atoms. The algorithm retrieves the Fourier atoms from these clean windows by applying Prony's method and obtains the spikes from the residual resulting from removing the Fourier atoms from the original signal. When the sparsity satisfies $K_f\,K_s < N/2$ the algorithm is guaranteed to succeed.

II. Noiseless average performance

The previous sparsity bound is a worst-case result since it is possible to find a counterexample that makes the algorithm fail when $K_f\,K_s = N/2$. However, the probability of occurrence of such counterexamples tend to zero when N is large. If this sparsity constraint is not satisfied, that is, when the product of the number of Fourier atoms and spikes goes beyond $N/2$, simulation results show that ProSparse is able to find the sparse solutions for a much larger area of the (K_s, K_f) plane. In fact, we are able to prove the existence of a phase transition phenomenon for the probability of success of ProSparse. For large N and the number of spikes given by $K_s = \alpha\,N$, the algorithm succeeds with high probability when the number of Fourier atoms is below a level that depends on α and fails with high probability above this level. This phase transition behaviour is illustrated in Figure 1, where the probability of success has been obtained empirically by simulating 100 different realizations of the sparse vector $\boldsymbol{x}$ for each pair of sparsity levels (K_f, K_s). Specifically, the following result can be shown:

Proposition 1. *Let $\boldsymbol{y} \in \mathbb{C}^N$ be a mixture of Fourier atoms and spikes chosen uniformly at random. Let $K_f = \tau \log N$ be the number of Fourier atoms. If there are $K_s = \alpha\,N$ spikes, then,*

$$\lim_{N\to\infty} \mathbb{P}\{\textit{algorithm succeeds}\} = \begin{cases} 0, & \textit{if } \tau > -1/\log(1-\alpha), \\ 1, & \textit{otherwise.} \end{cases}$$

III. Noisy algorithm

In the presence of noise, the search for the clean windows becomes unreliable. Thus, we apply a slightly different approach that is also based on Prony's method. When samples due to Fourier atoms are corrupted with additive noise, Prony's performance can be considerably improved by applying a denoising technique known as Cadzow [5]. The noisy version of ProSparse is therefore based on iteratively removing the K_s spikes by considering the entire signal $\boldsymbol{y}$ and assuming that the spikes are additive noise: the spikes are estimated from the residual between a denoised version of the signal and the actual noisy signal. After removing the spikes, the Fourier atoms are estimated applying Prony's method to the entire signal $\boldsymbol{y}$. When the dictionary is given by the union of bases, the same strategy can be applied to the Fourier transform of $\boldsymbol{y}$ where the spikes are estimated by removing the Fourier atoms items iteratively.

Figures 2 and 3 show simulation results where the probability of retrieving the correct support of the sparse vector are empirically obtained for the noisy ProSparse and Basis Pursuit Denoise (BPDN) algorithms. In the union of bases scenario, ProSparse outperforms BPDN by a small margin. This margin is significantly increased when we consider a Fourier frame.

IV. Conclusion

ProSparse is a polynomial complexity algorithm that is able to find *all* the sparse solutions that satisfy some sparsity conditions. The original paper [4] presents deterministic bounds when some sparsity constraint is satisifed. Here, we have shown that the algorithm is able to find the sparse solution with high probability well beyond these sparsity levels. A noisy sparse reconstruction is also presented that outperforms BPDN in a number of scenarios.

Acknowledgment

This work was supported by European Research Council (ERC) starting investigator award Nr. 277800 (RecoSamp).

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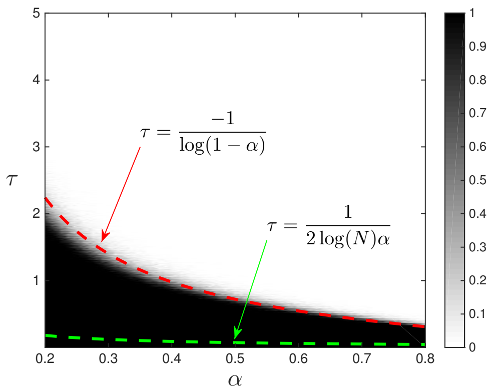
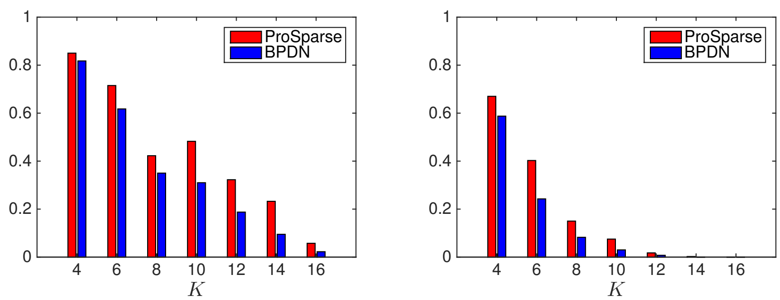


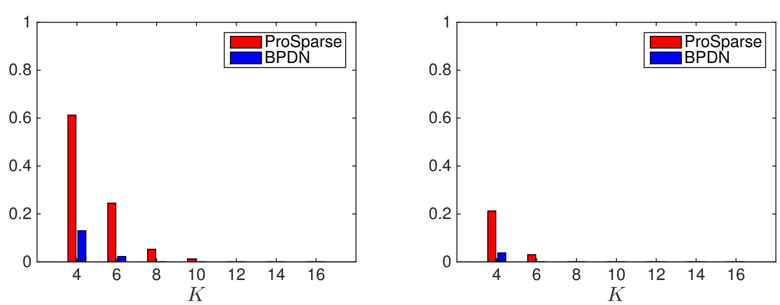
Fig. 1: Empirical probability of success of ProSparse and deterministic and probabilistic bounds when there are $K_s = \alpha N$ spikes and $K_f = \tau \log N$ Fourier atoms, with $N = 10^6$. In green, the deterministic bound, in red the phase transition bound.



(a) SNR = 10 dB

(b) SNR = 5 dB

Fig. 2: Empirical probability of support retrieval for different levels of noise when the dictionary is the union of Fourier and canonical bases. $N = 64$ and $K_s = K_f = 50\% K$. For each sparsity level, 20 different realizations of the sparse vector $\boldsymbol{x}$ have been generated and for each vector 20 different realizations of complex white Gaussian noise.



(a) SNR = 10 dB

(b) SNR = 5 dB

Fig. 3: Empirical probability of support retrieval for different levels of noise when the dictionary is the union of a Fourier frame and the canonical basis. The dimension of the Fourier frame is 64×256 and for the canonical basis $N = 64$. $K_s = 25\% K$ and $K_f = K - K_s$. For each sparsity level, 20 different realizations of the sparse vector $\boldsymbol{x}$ have been generated and for each vector 20 different realizations of complex white Gaussian noise.